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Bootstrap Co-integration Rank Testing: The Role of Deterministic Variables and Initial Values in the Bootstrap Recursion *

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Abstract

In this paper we investigate the role of deterministic components and initial values in bootstrap likelihood ratio type tests of co-integration rank. A number of bootstrap procedures have been proposed in the recent literature some of which include estimated deterministic components and non-zero initial values in the bootstrap recursion while others do the opposite. To date, however, there has not been a study into the relative performance of these two alternative approaches. In this paper we fill this gap in the literature and consider the impact of these choices on both OLS and GLS de-trended tests, in the case of the latter proposing a new bootstrap algorithm as part of our analysis. Overall, for OLS de-trended tests our findings suggest that it is preferable to take the computationally simpler approach of not including estimated deterministic components in the bootstrap recursion and setting the initial values of the bootstrap recursion to zero. For GLS de-trended tests, we find that the approach of Trenkler (2009), who includes a restricted estimate of the deterministic component in the bootstrap recursion, can improve finite sample behaviour further.

Keywords: Co-integration; trace tests; i.i.d. bootstrap; OLS and GLS de-trending.

J.E.L. Classifications: C30, C32.

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1 Introduction

Likelihood-based procedures for testing the co-integration rank in VAR systems of $I(1)$ variables, see Johansen (1996) and Saikkonen & Lütkepohl (2000), are extensively used in empirical research. However, it is now well understood that the finite sample properties of these tests, when based on asymptotic inference, can be quite poor; see, in particular, Johansen (2002) and the references therein. It is also well-known that the bootstrap, when correctly implemented, can be an important device to compute critical values of asymptotic tests in samples of finite size thereby delivering tests with empirical rejection frequencies closer to the nominal level. As a consequence, it is not surprising that there has been an increasing interest in using bootstrap methods in determining the co-integration rank in vector autoregressive models. For the use of bootstrap tests in co-integrated VAR models with independent and identically distributed (i.i.d.) innovations, see, in particular, Swensen (2006) and Trenkler (2009), while for VAR models with potentially heteroskedastic innovations, see Cavaliere, Rahbek & Taylor (2010).

In cases where deterministic components are allowed for, a key difference exists between the bootstrap recursion used to generate the bootstrap data in some of the approaches outlined above. In particular, the bootstrap recursion in Swensen (2006) includes an estimate of the deterministic component (obtained from estimating both the restricted and unrestricted VAR models) together with initial values taken as the corresponding initial values of the original data, while the corresponding recursion in Cavaliere et al. (2010) imposes zero start values and does not include an estimate of the deterministic component. Trenkler (2009) also includes an estimate of the deterministic component in his bootstrap recursion but, unlike Swensen (2006), his estimate is based only on the restricted VAR model. As argued in Cavaliere et al. (2010), the original statistics being bootstrapped are exact invariant to the deterministic component (by usual least squares projection arguments), and therefore these do not need to be included in the bootstrap recursion. Moreover, provided a constant is included in the deterministic component, the co-integration tests will be (exact) similar with respect to the starting values. In contrast we show that, crucially, the bootstrap tests obtained from the recursion in Swensen (2006) are *not* exact invariant to the level term in the deterministic component. In contrast, the tests outlined in Trenkler (2009), which include a restricted estimate of the deterministic component, are exact invariant to the level term.

In this paper our aim is to investigate which of these two approaches delivers the better finite sample performance. We investigate approaches based both on OLS de-trending and on GLS de-trending; for the former we use the trace tests of Johansen (1996) and for the

latter we use the corresponding trace tests of Saikkonen & Lütkepohl (2000). We find that the simpler recursion where no estimated deterministics are included and where initial values are set to zero delivers the best finite sample performance in the context of OLS de-trended tests, but is slightly inferior to the approach outlined in Trenkler (2009) for the case of GLS de-trending. The approach of Swensen (2006) displays the worst behaviour of all the approaches under both OLS and GLS de-trending with dependence of both size and power on the magnitude of the level of the deterministic component. We concentrate our attention in this paper on the trace type tests from a VAR model which contains a linear trend component. Qualitatively similar conclusions are drawn from Monte Carlo results in the case where the VAR contains only a constant and for inference based on the corresponding maximal eigenvalue type tests. These additional results are available from the authors on request.

The paper is organized as follows. Section 2 outlines our reference co-integrated VAR model and reviews the asymptotic likelihood-based trace tests of Johansen (1996) and Saikkonen & Lütkepohl (2000). In section 3 we outline the various bootstrap implementations of the trace tests from section 2, focusing on the issue of whether or not to include the estimated deterministic component in the bootstrap recursion and how to treat the initial values of the recursion. In section 4 we outline the invariance properties of the various statistics discussed in sections 2 and 3, showing that the approach of Swensen (2006) results in bootstrap tests which are not exact invariant to the level term in the deterministic component. In section 5 we compare the finite sample performance of both the asymptotic tests from section 2 and the various bootstrap tests outlined in section 3 for a variety of co-integrated models. Section 6 concludes.

In the following '$\xrightarrow{w}$' denotes weak convergence, '$\xrightarrow{p}$' convergence in probability, and '$\xrightarrow[p]{w}$' weak convergence in probability (Giné and Zinn, 1990; Hansen, 1996), in each case as the sample size diverges to positive infinity; $\mathbb{I}(\cdot)$ denotes the indicator function, and '$x := y$' (' $x =: y$') indicates that x is defined by y (y is defined by x). For any $m \times n$ matrix A, if A is of full column rank $n < m$, then $A_\perp$ denotes an $m \times (m-n)$ matrix of full column rank satisfying $A'_\perp A = 0$. For any square matrix, A, $|A|$ is used to denote the determinant of A.

2 The Model Framework and Rank Tests

Let us consider a n-dimensional times series $y_t := (y_{1t}, \ldots, y_{nt})'$, $t = 1, \ldots, T$, which is generated by

$$y_t = \mu_0 + \mu_1 t + x_t, \quad t = 1, 2, \ldots, \tag{2.1}$$

where μ_0 (the level term) and μ_1 (the linear trend coefficient) are unknown $(n \times 1)$ parameter vectors. Hence, the deterministic part consists of a constant and a linear trend. The term x_t is an unobservable stochastic error process which is assumed to follow a vector autoregressive process of order p, VAR(p),

$$x_t = A_1 x_{t-1} + \cdots + A_p x_{t-p} + \varepsilon_t, \quad t = 1, 2, \ldots, \tag{2.2}$$

where A_j, $j = 1, \ldots, p$, are $(n \times n)$ coefficient matrices and the initial values, $x_t = 0$, $t \leq 0$, are taken to be fixed.

As usual, we can write the VAR(p) model in vector error correction (VEC) form as

$$\Delta x_t = \Pi x_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta x_{t-j} + \varepsilon_t, \quad t = 1, 2, \ldots, \tag{2.3}$$

where Γ_j, $j = 1, \ldots, p-1$, are $(n \times n)$ lag coefficient matrices and the impact matrix Π satisfies $\Pi = \alpha\beta'$, where α and β are full column $(n \times r)$ matrices for $0 < r \leq n$.

Multiplying (2.1) by $A(L) := I_n - A_1 L - \cdots - A_p L^p = I_n \Delta - \alpha\beta' L - \Gamma_1 \Delta L - \cdots - \Gamma_{p-1} \Delta L^{p-1}$ and rearranging yields the VECM representation for y_t

$$\Delta y_t = \nu + \alpha(\beta' y_{t-1} - \phi(t-1)) + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + \varepsilon_t, \quad t = p+1, p+2, \ldots, \tag{2.4}$$

where $\nu := -\Pi\mu_0 + \Gamma\mu_1 = -\alpha\theta + \Gamma\mu_1$, $\Gamma := I_n - \sum_{j=1}^{p-1} \Gamma_j$, $\theta := \beta'\mu_0$, and $\phi := \beta'\mu_1$.

Throughout the paper, the process in (2.3) is assumed to satisfy the following assumptions.

Assumption 1: (a) *All of the characteristic roots associated with* (2.3)*, that is the solutions to the characteristic equation* $A(z) := (1-z) I_n - \alpha\beta' z - \Gamma_1 z (1-z) - \cdots - \Gamma_{p-1} z^{p-1} (1-z) = 0$*, lie either outside the unit circle or are equal to unity;* (b) $|\alpha'_\perp \Gamma \beta_\perp| \neq 0$*, with* $\Gamma := I_n - \Gamma_1 - \cdots - \Gamma_{p-1}$.

Assumption 2: *The innovations* $\{\varepsilon_t\}$ *form a martingale difference sequence with respect to the filtration* $\mathcal{F}_t$*, where* $\mathcal{F}_{t-1} \subseteq \mathcal{F}_t$ *for* $t = ..., -1, 0, 1, 2, ...$*, satisfying: (i) the global homoskedasticity condition:*

$$\frac{1}{T} \sum_{t=1}^{T} \mathrm{E}\left(\varepsilon_t \varepsilon_t' | \mathcal{F}_{t-1}\right) \xrightarrow{p} \Sigma > 0, \tag{2.5}$$

and (ii) $\mathrm{E} \left\|\varepsilon_t\right\|^4 \leq K < \infty$.

Assumption 1 is standard in the co-integration testing literature, while Assumption 2, which is used by Cavaliere et al. (2010), implies that ε_t is a serially uncorrelated,

potentially conditionally heteroskedastic process. The latter therefore contrasts with the assumption that ε_t is i.i.d. as made in Johansen (1996) and Swensen (2006).

We consider the so-called trace test version, i.e. we aim to test the pair of hypotheses

$$H_0(r) : \text{rk}(\Pi) = r \quad vs. \quad H_1(r) : \text{rk}(\Pi) = n. \tag{2.6}$$

For unknown parameters α, β, $\Psi := (\Gamma_1, ..., \Gamma_{p-1})$, ν, ϕ, and when α and β are $p \times r$ matrices, not necessarily of full rank, (2.4) denotes our co-integrated VAR model for the observable y_t. The model may then be written in the compact form

$$Z_{0t} = \alpha\beta^{+\prime} Z_{1t} + \delta Z_{2t} + \varepsilon_t \tag{2.7}$$

with $Z_{0t} := \Delta y_t$, $Z_{1t} := (y'_{t-1}, t)'$, $Z_{2t} := (U'_t, 1)'$, where $U_t := \left(\Delta y'_{t-1}, ..., \Delta y'_{t-p+1}\right)'$, $\beta^+ := (\beta', -\phi')'$ and $\delta := (\Psi, \nu)$.

As is standard, let $M_{ij} := T^{-1} \sum_{t=1}^{T} Z_{it} Z'_{jt}$, $i, j = 0, 1, 2$, with Z_{it} defined as in (2.7), and let $S_{ij} := M_{ij} - M_{i2} M_{22}^{-1} M_{2j}$, $i, j = 0, 1$. Solving $\left|\lambda S_{11} - S_{10} S_{00}^{-1} S_{01}\right| = 0$, we obtain the ordered generalized eigenvalues $\hat{\lambda}_1 > \cdots > \hat{\lambda}_n$. The (pseudo) likelihood ratio [PLR] test for the pair of hypotheses in (2.6) then rejects for large values of the trace statistic[1]

$$LR_r := -T \sum_{i=r+1}^{n} \log(1 - \hat{\lambda}_i).$$

Cavaliere et al. (2010) derive the limiting null distribution for LR_r for data generated according to (2.2) under Assumptions 1 and 2, and this result is reproduced for convenience in the following theorem.

Theorem 1 *Let x_t be generated as in (2.2) under Assumptions 1 and 2. Then under $H_0(r)$,*

$$LR_r \xrightarrow{w} \mathsf{tr}(\mathcal{LR}_{r,B}) =: LR_{r,\infty} \tag{2.8}$$

where

$$\mathcal{LR}_{r,B} := \int_0^1 (dB_{n-r}(u)) F_{n-r}(u)' \left(\int_0^1 F_{n-r}(u) F_{n-r}(u)' du \right)^{-1} \int_0^1 F_{n-r}(u) (dB_{n-r}(u))' \tag{2.9}$$

with $B_{n-r}(\cdot)$ a $(n-r)$-variate standard Brownian motion and $F_{n-r} := (B'_{n-r}, u|1)'$, where the notation $a|b := a(\cdot) - \int a(s)b(s)'ds (\int b(s)b(s)'ds)^{-1} b(\cdot)$ denotes the projection residuals of a onto b.

[1]Notice that the subscript r in LR_r is a generic notation denoting the null rank being tested. If the specific rank $r = 0$ were being tested, for example, then the statistic would be referred to as LR_0. The same convention will be adopted for all other statistics introduced in this paper.

The PLR test, LR_r, outlined above is based on OLS de-trending. More recently, Saikkonen & Lütkepohl (2000) have suggested alternative tests, in the same spirit as Elliott *et al.* (1996) for univariate unit root tests, which use (pseudo) GLS de-trending to adjust the data for deterministic terms before applying the LR test procedure described above. To that end, define

$$a_{0t} := \begin{cases} 1 & \text{for} \quad t \geq 1 \\ 0 & \text{for} \quad t \leq 0 \end{cases} \qquad \text{and} \qquad a_{1t} = \begin{cases} t & \text{for} \quad t \geq 1 \\ 0 & \text{for} \quad t \leq 0 \end{cases}$$

and multiplying (2.1) by $A(L)$ we obtain

$$A(L)y_t = H_{0t}\mu_0 + H_{1t}\mu_1 + \varepsilon_t, \qquad t = 1, \ldots, T, \tag{2.10}$$

where $H_{it} := A(L)a_{it}$ $(i = 0, 1)$, and $\varepsilon_t = A(L)x_t$. Furthermore, define Q such that QQ'^{-1} and multiply (2.10) by Q', then we have

$$Q'A(L)y_t = Q'H_{0t}\mu_0 + Q'H_{1t}\mu_1 + \eta_t, \qquad t = 1, \ldots, T, \tag{2.11}$$

where $\eta_t := Q'\varepsilon_t$. The matrix Q can be chosen as $Q = [\Omega^{-1}\alpha(\alpha'^{-1}\alpha)^{-1/2} : \alpha_\perp(\alpha_\perp'^{-1}\alpha_\perp)^{-1/2}]$. Hence, the error term η_t has a zero mean and a unit covariance matrix as it is required for a GLS transformation. To render the GLS estimation feasible, Saikkonen & Lütkepohl (2000) propose using the reduced rank (RR) estimators $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\Gamma}_j$, $j = 1, \ldots, p-1$, and $\widetilde{\Omega}$ which are obtained from (2.4) by applying r, the rank under the null hypothesis of the co-integration test. Based on these RR estimators, one can compute the estimators $\tilde{Q}$ and $\tilde{H}_{it}$ $(i = 0, 1)$. Thus, feasible GLS estimators of μ_0 and μ_1, say $\tilde{\mu}_0$ and $\tilde{\mu}_1$, are then obtained by multivariate LS estimation of the model

$$\tilde{Q}'\tilde{A}(L)y_t = \tilde{Q}'\tilde{H}_{0t}\mu_0 + \tilde{Q}'\tilde{H}_{1t}\mu_1 + \tilde{\eta}_t, \qquad t = 1, \ldots, T. \tag{2.12}$$

The estimated deterministic terms are used to adjust y_t, giving the GLS de-trended analogue $\tilde{x}_t := y_t - \tilde{\mu}_0 - \tilde{\mu}_1 t$ of x_t. Then, the LR-type test is performed with respect to the model

$$\Delta\tilde{x}_t = \Pi\tilde{x}_{t-1} + \sum_{j=1}^{p-1}\Gamma_j\Delta\tilde{x}_{t-j} + \tilde{\varepsilon}_t, \quad t = p+1, \ldots, T, \tag{2.13}$$

where $\tilde{\varepsilon}_t$ is an error term, specifically the GLS de-trended analogue of ε_t. Since $\tilde{x}_t$ is adjusted by the deterministic terms, a test version without deterministic terms, as in Johansen (1988), is applied. Denote the ordered generalized eigenvalues obtained from the corresponding eigenvalue problem by $\tilde{\lambda}_1 > \cdots > \tilde{\lambda}_n$. Then, the GLS de-trended trace

test for the pair of hypotheses in (2.6) rejects for large values of the statistic[2]

$$GLS_r := -T \sum_{i=r+1}^{n} \log(1 - \tilde{\lambda}_i). \tag{2.14}$$

The limiting null distribution of GLS_r is given in the following theorem, critical values from which can be computed using the response surface approach of Trenkler (2008). This result was derived under a slightly stronger version of Assumption 2 by Saikkonen & Lütkepohl (2000); the generalisation to cover Assumption 2 is entirely straightforward given the results in Saikkonen & Lütkepohl (2000) and is therefore omitted.

Theorem 2 *Let the conditions of Theorem 1 hold. Then under* $H_0(r)$*,*

$$GLS_r \xrightarrow{w} \mathsf{tr}(\mathcal{GLS}_{r,B}) =: GLS_{r,\infty} \tag{2.15}$$

where

$$\mathcal{GLS}_{r,B} := \int_0^1 dB^*_{n-r}(u) B^*_{n-r}(u)' \left(\int_0^1 B^*_{n-r}(u) B^*_{n-r}(u)' ds \right)^{-1} \left(\int_0^1 B^*_{n-r}(u) dB^*_{n-r}(u)' \right) \tag{2.16}$$

with $B^*_{n-r}(u) := B_{n-r}(u) - uB_{n-r}(1)$ *is a* $(n-r)$*-variate Brownian bridge.*

Remark 2.1. As stated in Theorem 1 of Saikkonen & Lütkepohl (2000), the level parameter μ_0 is not consistently estimated in the direction of $\beta_\perp$ because μ_0 is not identified in that direction in (2.4). Therefore, Saikkonen, Lütkepohl & Trenkler (2006) suggest avoiding the estimation of the level parameter in the first stage and only adjusting for the trend component. However, the relative performance of the resulting bootstrap tests with respect to the asymptotic test is little different from Johansen's PLR test; see Trenkler (2009) for further discussion on this point. Consequently, we will not consider this version of the test any further here and will focus on the GLS_r test procedure as outlined above.

3 Bootstrap Co-integration Tests

In this section we describe the various bootstrap schemes which we will subsequently compare in section 4. In Algorithm 1, we first outline our leading recursive bootstrap used to generate the so-called pseudo or bootstrapped observations, $y_1^*, \ldots, y_T^*$. In the case of the OLS de-trended PLR test, LR_r, this algorithm has been previously suggested in

[2]Note that the generalized eigenvalue problem described here is slightly different from the one in Saikkonen & Lütkepohl (2000). However, the eigenvalue problems can be transformed into each other by an appropriate redefinition of the respective eigenvalues.

Cavaliere et al. (2010), and has the key property that both deterministic terms and initial values are set to zero in the bootstrap data recursion. This scheme can be regarded as an adjusted version of the bootstrap algorithms discussed in Swensen (2006) and Trenkler (2009), which, in contrast, do include estimated deterministic components and non-zero initial values in the bootstrap recursion. These alternative algorithms are subsequently discussed in Remarks 3.3 and 3.4.

Algorithm 1.

(1) Estimate an unrestricted version of (2.4) setting $r = n$, i.e. estimate a VAR(p) model for y_t, in order to obtain the OLS estimators $\hat{\Gamma}_j$, $j = 1, \ldots, p-1$, and the ordinary least squares (OLS) residuals $\hat{\varepsilon}_{p+1}, \ldots, \hat{\varepsilon}_T$.

(2) The remaining parameters are estimated by performing a RR regression of (2.4) under the rank hypothesis $H_0(r) : \text{rk}(\Pi) = r$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the estimators of α and β respectively.

(3) Construct the bootstrap sample data, y_t^*, $t = p, \ldots, T$, recursively from

$$\Delta y_t^* = \tilde{\alpha}\tilde{\beta}' y_{t-1}^* + \sum_{j=1}^{p-1} \hat{\Gamma}_j \Delta y_{t-j}^* + \varepsilon_t^*, \tag{3.1}$$

with sampled residuals ε_t^* drawn with replacement from the estimated residuals $\hat{\varepsilon}_{p+1}, \ldots, \hat{\varepsilon}_T$. The starting values of the recursion, $y_1^*, \ldots, y_p^*$, are set equal to 0.

(4) Obtain the bootstrap test statistics, LR_r^{1*} and GLS_r^{1*}, analogous to LR_r and GLS_r, respectively.

(5) Bootstrap p-values are then computed as, $p_{r,T}^*(LR_r) := 1 - G_{LR,r,T}^*(LR_r)$ and $p_{r,T}^*(GLS_r) := 1 - G_{GLS,r,T}^*(GLS_r)$, where $G_{LR,r,T}^*(\cdot)$ and $G_{GLS,r,T}^*(\cdot)$ denote the conditional (on the original data) cumulative distribution functions (cdf) of LR_r^{1*} and GLS_r^{1*}, respectively.

Remark 3.1. Cavaliere et al. (2010) establish the asymptotic validity of the test based on the bootstrap PLR statistic, LR_r^{1*}, from Algorithm 1. In particular they demonstrate that the bootstrap LR_r^{1*} statistic attains the same first-order limiting null distribution as the LR_r statistic. Formally, they show that under the conditions of Theorem 1, $LR_r^{1*} \xrightarrow{w}_p LR_{r,\infty}$ and that, as a consequence, the associated bootstrap p-value, $p_{r,T}^*(LR_r)$ is (asymptotically) uniformly distributed under the null hypothesis, leading to tests with (asymptotically) correct size. It is entirely straightforward but tedious to show using the

results in Cavaliere et al. (2010), that similar conclusions hold for the GLS de-trended statistic, GLS_r, and its bootstrap analogue. Specifically, under the conditions of Theorem 1, $GLS_r^{1*} \xrightarrow{w}_p GLS_{r,\infty}$ and, consequently, $p^*_{r,T}(GLS_r)$ is (asymptotically) uniformly distributed under the null hypothesis.

Remark 3.2. As discussed in Cavaliere et al. (2010), the unknown cdfs, $G^*_{LR,r,T}(\cdot)$ and $G^*_{GLS,r,T}(\cdot)$, required in Step 5 of Algorithm 1 can be estimated through numerical simulation. This is done by generating B (conditionally) independent bootstrap statistics, $LR^{1*}_{r,b}$ and $GLS^{1*}_{r,b}$, $b = 1, ..., B$, and computing the estimated bootstrap p-values of the tests as

$$\tilde{p}^*_{r,T}(LR_r) := \frac{1}{B}\sum_{b=1}^{B} \mathbb{I}(LR^{1*}_{r,b} > LR_r) \tag{3.2}$$

$$\tilde{p}^*_{r,T}(GLS_r) := \frac{1}{B}\sum_{b=1}^{B} \mathbb{I}(GLS^{1*}_{r,b} > GLS_r). \tag{3.3}$$

For $B \to \infty$, we have that $\tilde{p}^*_{r,T}(LR_r) \xrightarrow{a.s.} p^*_{r,T}(LR_r)$ and $\tilde{p}^*_{r,T}(GLS_r) \xrightarrow{a.s.} p^*_{r,T}(GLS_r)$; see e.g. Hansen (1996). Estimated p-values for the alternative bootstrap procedures discussed below can be obtained in the same way.

Remark 3.3. In terms of the OLS de-trended LR_r test, Algorithm 1 above is similar to Algorithm 1 in Swensen (2006) except that a different recursion is used in step 3. Specifically, Algorithm 1 in Swensen (2006) replaces the bootstrap recursion in (3.1) with

$$\Delta y^*_t = \hat{\nu} + \tilde{\alpha}(\tilde{\beta}' y^*_{t-1} - \tilde{\phi}(t-1)) + \sum_{j=1}^{p-1} \hat{\Gamma}_j \Delta y^*_{t-j} + \varepsilon^*_t$$

where $\hat{\nu}$ is the OLS estimator of ν obtained from an unrestricted VAR(p) model for y_t and $\tilde{\phi}$ is the RR estimator of ϕ obtained from an estimation of (2.4) under the rank hypothesis $H_0(r)$. The starting values of the recursion, $y^*_1, \ldots, y^*_p$, are set equal to $y_1, \ldots, y_p$. Thus, parameter estimates obtained from two different models are combined within the bootstrap recursion which, unlike Algorithm 1 of this paper, contains an estimate of the deterministic component. Trenkler (2009) argues that, specifically, the combination of $\hat{\nu}$ with the VECM estimates $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\phi}$ can cause serious deteriorations in the small sample properties of the bootstrap tests, in particular of the GLS test. In fact, his simulation results show that the larger the deterministic component of the data generating process [DGP] is, the smaller are the tests' rejection frequencies. Hence, serious under-rejection and poor power properties have to be expected. We will analyze in our simulation experiment whether setting the deterministic part and the initial values to zero, as in Algorithm 1 of this paper, avoids these distortions. To this end, we also consider Algorithm 1 of

Swensen (2006) which we will refer to as Algorithm 1d in what follows. The resulting OLS and GLS de-trended bootstrap test statistics are labelled LR_r^{1d*} and GLS_r^{1d*}, respectively.

Remark 3.4. Trenkler (2009) suggests replacing the bootstrap recursion in (3.1) with

$$\Delta y_t^* = \tilde{\nu} + \tilde{\alpha}(\tilde{\beta}' y_{t-1}^* - \tilde{\phi}(t-1)) + \sum_{j=1}^{p-1} \tilde{\Gamma}_j \Delta y_{t-j}^* + \varepsilon_t^* \tag{3.4}$$

where $\tilde{\nu}$ and $\tilde{\Gamma}_j$, $j = 1, \ldots, p-1$ are RR estimators obtained from an estimation of (2.4) under the rank hypothesis $H_0(r)$. Hence, and in contrast to Algorithm 1 of Swensen (2006), estimators from different models are not combined. Simulation results in Trenkler (2009) suggest that the resulting bootstrap tests are significantly less distorted than the corresponding tests based on Algorithm 1 of Swensen (2006) in the case of a non-zero deterministic component in the DGP. As a consequence, we will include this scheme in our comparative study in section 4. We will consider both the original version considered in Trenkler (2009) with estimated deterministic component and nonzero initial values (which we denote by Algorithm 2d in what follows), as well as a new version based on the recursion $\Delta y_t^* = \tilde{\alpha}\tilde{\beta}' y_{t-1}^* + \sum_{j=1}^{p-1} \tilde{\Gamma}_j \Delta y_{t-j}^* + \varepsilon_t^*$ with $y_1^*, \ldots, y_p^*$ set equal to 0 (referred to as Algorithm 2 in what follows). We denote the resulting OLS and GLS de-trended bootstrap test statistics from Algorithm 2d as LR_r^{2d*} and GLS_r^{2d*}, respectively, and those from Algorithm 2 as LR_r^{2*} and GLS_r^{2*}, respectively.

Remark 3.5. All of the bootstrap algorithms discussed above require that the roots of the equation $|\hat{A}^*(z)| = 0$ are either one or are outside the unit circle, where

$$\hat{A}^*(z) := (1-z) I_p - \hat{\alpha}\hat{\beta}' z - \hat{\Gamma}_1 (1-z) z - ... - \hat{\Gamma}_{k-1} (1-z) z^{k-1}.$$

Moreover, it is also required that $|\hat{\alpha}_\perp' \hat{\Gamma} \hat{\beta}_\perp| \neq 0$ where $\hat{\Gamma} := I_p - \hat{\Gamma}_1 - ... - \hat{\Gamma}_{k-1}$. While the latter condition is always satisfied in practice, if the former condition is not met, then the bootstrap algorithms cannot be implemented, since the bootstrap samples may become explosive; cf. Swensen (2006, Remark 1). As a consequence, we will not consider iterations in the simulations where the former condition is violated.

4 Invariance Issues

All of the tests considered are exact invariant to the trend coefficient, μ_1. However, while all of the tests considered in sections 2 and 3 are asymptotically invariant to the level parameter, μ_0, not all of the tests considered in this paper are exact invariant to μ_0. Both the LR_r and GLS_r statistics are invariant to the value of μ_0 given that they are

both based on de-trended data. Moreover, the bootstrap statistics LR_r^{1*} GLS_r^{1*}, LR_r^{2*}, and GLS_r^{2*} are also invariant to μ_0 since the corresponding bootstrap algorithms do not contain any deterministic terms and because the initial values are set to zero.

For the remaining tests we note first that changes in μ_0 will only affect the estimates of the constant term vector ν in the *unrestricted* VAR model and in the VECM with the rank under H_0 imposed. The other estimates considered will not respond to variations in μ_0. If μ_0 is varied, say by $\nabla\mu_0$, the estimates of the constant term vector ν in the unrestricted VAR model and the restricted VECM adjust in such a way that the variation in μ_0 is exactly matched. Note, however, that these adjustments in $\hat{\nu}$ and $\tilde{\nu}$ depend on the model setup used for estimating the model parameters. Accordingly, all bootstrap data are only shifted exactly by $\nabla\mu_0$ if the *same* model is used to estimate ν and to generate the bootstrap data. This is indeed the case for the recursion of bootstrap Algorithm 2d. Here, the VECM estimated under the rank null hypothesis is used for both the estimation of all parameters and the generation of the data. Thus, LR_r^{2d*} and GLS_r^{2d*} also do not depend on μ_0 since the test procedures are invariant to simple shifts in the data.

The latter result does not hold in general for LR_r^{1d*}, and GLS_r^{1d*}. The reason is that the VECM-recursion in Algorithm 1d uses some parameter estimates (including the estimate of ν) obtained from a different model, specifically the unrestricted VAR model. Hence, the change in $\hat{\nu}$ cannot exactly reproduce $\nabla\mu_0$. To be more precise, the initial values will, by definition, change by $\nabla\mu_0$ but the same will not happen with the observations of the remaining bootstrap sample. In fact, the inappropriate reproduction of $\nabla\mu_0$ will generate a mismatch between initial values and the remainder of the bootstrap sample. This mismatch will depend on the underlying magnitude of μ_0. We will analyze in the simulation study how this mismatch affects the small sample properties of LR_r^{1d*} and GLS_r^{1d*}.

There is, however, one special case where LR_r^{1d*} and GLS_r^{1d*} are invariant to μ_0: if the null $r = 0$ is tested within a VAR(1). In this setup, the recursion in Algorithm 1d is equal to $\Delta y_t^* = \hat{\nu} + \varepsilon_t^*$. Here, y_1^* will exactly change by $\nabla\mu_0$ and y_t^*, $t = 2, 3, \ldots, T$, will change by $\nabla\mu_0 + (t-1)\nabla\hat{\nu}$. Hence, the change is described by a linear trend and can, therefore, be perfectly captured by both the OLS and GLS de-trended tests. Hence, LR_0^{1d*} and GLS_0^{1d*} do not depend on μ_0 in this setup. Note, however, that this result does not carry over to higher order VAR processes. This is due to fact that all initial values $y_1^*, \ldots, y_p^*$ will adjust by $\nabla\mu_0$ so that the change in the bootstrap sample is not a simple linear trend. In fact, the change may even be manifested as a higher order trend. In any case, we again have a mismatch between initial values and the remainder of the bootstrap sample.

5 Numerical Results

In this section we use Monte Carlo simulation methods to compare the finite sample size and power properties of the various bootstrap and asymptotic tests discussed previously in Section 3.

We first consider a data-based DGP by referring to an empirical study of King, Plosser, Stock & Watson (1991) (KPSW) who analyse a small macroeconomic model for the U.S. which consists of the logarithms of per-capita private real GNP, per-capita real consumption, and per-capita gross private domestic fixed investment. We estimate a subset-VECM with one lag and two restricted co-integrating relationships using quarterly data in logarithms for the period 1949:1-1988:4. Subset restrictions have been imposed by using a *Top-Down* strategy employing the AIC.[3] We obtain the following process, which we refer to as the KPSW-DGP in what follows:

$$\Delta y_t = \begin{bmatrix} -0.038 \\ -0.186 \\ 0.032 \end{bmatrix} + \begin{bmatrix} 0 & -0.026 \\ 0.217 & -0.150 \\ 0.126 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} y_{t-1} + \begin{bmatrix} 0 & 0 & 0.154 \\ 0 & 0.282 & 0.660 \\ 0.272 & 0.162 & 0 \end{bmatrix} \Delta y_{t-1} + \epsilon_t \tag{5.1}$$

where $\epsilon_t \sim$ i.i.d. $N(0, \Sigma)$ with

$$\Sigma = 10^{-4} \begin{bmatrix} 0.588 & 0.821 & 0.465 \\ & 4.870 & 1.688 \\ & & 1.376 \end{bmatrix}.$$

As starting values we chose the corresponding empirical data. The same process was used by Trenkler (2009) in a related simulation study. The results in Trenkler (2009) highlighted poor finite sample behaviour in both the asymptotic tests and the bootstrap co-integration tests from Remark 3.4 using the recursive scheme in (3.4) for this process. Hence, we may regard the KPSW-DGP as a demanding reference model for the bootstrap test procedures under consideration.

The second DGP we consider is a co-integrated VAR(2) process; cf. Johansen (2002) and Swensen (2006). We will consider processes of dimension n for $n = 2, ..., 5$. We set the true co-integrating rank, r_0 say, equal to one throughout. The general model we consider is therefore given by

$$y_t = \mu_0 + \mu_1 t + x_t \tag{5.2a}$$

$$\Delta x_t = \alpha\beta' x_{t-1} + \Gamma_1 \Delta x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, I_n), \quad t = 1, ..., T, \tag{5.2b}$$

[3]The subset restrictions have been obtained using JMulTi; see Lütkepohl & Krätzig (2004, Ch. 3).

where α and β are $n \times 1$ vectors and $\Gamma_1 = \xi I_n$ with $-1 < \xi < 1$. Following Johansen (2002) and Swensen (2006), we consider the parameter combinations, $\beta = (1, 0, ..., 0)'$ and $\alpha = (a_1, a_2, 0, ..., 0)'$, leading to the model

$$\Delta x_{1,t} = a_1 x_{1,t-1} + \Gamma_1 \Delta x_{t-1} + \varepsilon_{1,t} \tag{5.3a}$$

$$\Delta x_{2,t} = a_2 x_{1,t-1} + \Gamma_1 \Delta x_{t-1} + \varepsilon_{2,t} \tag{5.3b}$$

$$\Delta x_{i,t} = \Gamma_1 \Delta x_{t-1} + \varepsilon_{i,t}, \qquad i = 3, ..., n. \tag{5.3c}$$

We focus on $\xi = 0.5$ and consider three cases for a_1 and a_2: $a_1 = a_2 = -0.4$ (Case 1), $a_1 = a_2 = -0.1$ (Case 2), and $a_1 = a_2 = -0.8$ (Case 3). As described in Johansen (2002), Case 2 refers to a situation close to no co-integration whereas Case 3 represents a setup close to the case of $I(2)$ data. We do not consider VAR(1) processes since they are of limited interest, given the discussion in section 4 on the invariance properties of the test statistics. Moreover, due to the exact invariance properties of the tests discussed in section 4, we may set $\mu_1 = 0$ in all simulations with no loss of generality.

The specific values of μ_0 in the KPSW-DGP are implied by the set-up in (5.1). To this end, we have to recover μ_0 in the generic formulation of the model in (2.1) from the parameters of the DGP in (5.1). However, as pointed out in Trenkler (2009), only the differences of the entries in μ_0 are identified in (5.1) via $\beta'\mu_0 = (\alpha'\alpha)^{-1}\alpha'(\Gamma\mu_1 - \nu)$. The corresponding results in Trenkler (2009) show that these level differences amount to 20 to 200 times the standard deviation of the error terms in (5.1). Regarding the second simulation DGP considered in (5.2), since the error terms in (5.2b) have an identity covariance matrix, we set μ_0 equal to a unit vector multiplied by $0, 5, 20, 50, 200$, and 1000 in our simulations, where the first value represents the benchmark case of a zero level term. We do not present results for vectors μ_0 with different individual entries because such set-ups do not provide any new insights. Moreover, in order to eliminate the effect of the initial values we generate 200 pre-sample values initiated at the zero vector, in addition to the main sample of size T. The pre-sample values are discarded afterwards. Results are reported for $T = 50$, $T = 100$, and $T = 200$.

All tests are run at the 5% significance level. The rejection frequencies of the asymptotic Johansen and GLS tests are based on asymptotic critical values computed from response surfaces given in Doornik (1998) and Trenkler (2008), respectively. We use the response surface critical values since these are known to deliver a more accurate approximation of the tests' limiting distributions than the standard tabulated asymptotic critical value; see Doornik (1998). The computations are performed using the RNDNS function (with fixed seed) of GAUSS 9.0 for Windows. The number of replications is $R = 5000$. For determining the quantiles of the empirical bootstrap distributions we use $B = 499$

bootstrap replications.

5.1 KPSW-DGP

The results for the KPSW-DGP (5.1) are shown in Table 1. Looking at the results in Panel A, which concerns tests for the true null hypothesis that the co-integrating rank is two, it is immediately observed that the bootstrap tests based on Algorithm 1d of Swensen (2006) display poor empirical size properties. In particular, LR_2^{1d*} is far too conservative while GLS_2^{1d*} is badly over-sized, displaying worse size control even than the asymptotic LR_2 and GLS_2 tests, respectively. In contrast, using Algorithm 1, where the starting values and deterministic components are set to zero, yields significant improvements, most notably for smaller samples sizes, in empirical size *vis-à-vis* the asymptotic tests. In particular, both LR_2^{1*} and GLS_2^{1*} display a lesser degree of downward size distortion than LR_2 and GLS_2 respectively.

The poor finite sample properties of the GLS_2^{1d*} test have previously been discussed in Trenkler (2009), who shows that the unrestricted estimates of ν obtained from KPSW-DGP can be heavily biased in small samples. As a consequence, the bootstrap data generated by Algorithm 1d typically contain deviating drifts such that there is a mismatch between the initial values taken from the original data and the actual properties of the bootstrap data generated from Algorithm 1d (which use the estimated deterministic component). Moreover, the GLS-type tests tend to be sensitive to the initial values of the process, since these have a strong impact on the estimation of μ_0 in (2.1). Given the mismatch between the initial values and the remainder of the bootstrap sample, the GLS-adjustment for deterministic terms can therefore be quite inaccurate, resulting in poor small sample properties for GLS_2^{1d*}.

Using only parameter estimates from a restricted VECM within the bootstrap algorithm, as suggested by Trenkler (2009), is also beneficial relative to Algorithm 1d. However, LR_2^{1*} is still the best procedure among the OLS de-trended tests, although the difference relative to LR_2^{2*} is rather small. In contrast, GLS_2^{2d*}, the test originally proposed by Trenkler (2009), outperforms the other GLS de-trended tests. Hence, in the case of the GLS-type tests, setting the initial values and deterministic components to zero is beneficial when using Algorithm 1, but not when using Algorithm 2.

Overall there appears to be little difference between Algorithms 1 and 2. To be precise, once initial values and deterministic components are set to zero it does not seem to matter much whether the estimates for Ψ are taken from an estimated unrestricted VAR (Algorithm 1) or a restricted VECM (Algorithm 2). This result may well be attributable to the fact that the data obtained from Algorithms 1 and 2 are strongly correlated. On average

this correlation turns out to be about 0.8. In contrast, regarding Algorithms 2 and 2d on the one hand, and Algorithms 1 and 2d on the other, the correlations amount to about 0.6 and 0.7 (after trend adjustments), respectively. Moreover, the correlations involving Algorithm 1d are practically zero after trend adjustment. These smaller correlations help explain the differing performances of some of the bootstrap tests. Similar results are found for the VAR(2) processes which we analyze in detail in the next subsection.

Turning to the results for power (that is for tests of either the null hypothesis of rank one or zero), we see that bootstrapping tends to be associated with some loss of power. This effect is more pronounced when $r = 0$ is being tested. For the tests of $r = 1$ power losses are very mild. Finally, we observe that GLS_0^{1d*} displays a very large loss in power relative to both the asymptotic test, GLS_0, and the other GLS de-trended bootstrap tests. This, coupled with the bad over-size problems noted above, arguably renders the GLS_r^{1d*} test unusable in practice.

5.2 Co-integrated VAR(2)

We now discuss the results for the co-integrated VAR(2) processes (5.2) which are reported in Tables 2 - 9. Recalling that the true co-integrating rank, r_0, is one, we first comment on the empirical sizes of tests for the null hypothesis of $r = 1$ and then subsequently investigate the power properties of tests for $r = 0$. Notice that Tables 3, 5, 7, and 9 only report results for the LR_r^{1d*} and GLS_r^{1d*} tests since all the other tests are invariant to the value of μ_0; cf. section 4.

As was also observed for the results for KPSW-DGP, using Algorithm 1 instead of Algorithm 1d (i.e. leaving out the deterministic terms and using zero initial values) is clearly beneficial relative to the asymptotic tests. Moreover, applying LR_1^{1*} typically results in empirical sizes which are closer to the nominal level than those of LR_1^{1d*}, the latter tending to be under-sized, even when $\mu_0 = 0$. Other things being equal, the under-sizing seen in LR_1^{1d*} becomes more pronounced: (a) the larger is μ_0; (b) the smaller is the sample size, and (c) the larger the dimension of the system, n. For example, while LR_1^{1d*} has almost correct nominal empirical size for $T = 200$, $n = 2$ and $\mu_0 = 0$ under Case 2, for $T = 50$, $n = 5$ and $\mu_0 \geq 20$ empirical size drops below 1%. In contrast, recall from section 4 that LR_1^{1*} is invariant to the value of μ_0. Notice, however, that LR_1^{1*} has a tendency to be somewhat over-sized under Case 3 for $T = 50$ and $n > 2$. However, in these cases it still constitutes a massive improvement over the asymptotic test.

Similarly, GLS_1^{1*} turns out to be the better choice than GLS_1^{1d*}. In particular, GLS_1^{1*} avoids the excessive size distortions seen with GLS_1^{1d*} for $n = 2$ and reduces the often very large downward bias (in both size and power) of GLS_1^{1d*} for larger systems, in particular

for $T = 50$. Comparing the results in Tables 3, 5, 7, and 9, it is also observed that GLS_1^{1d*} is much more negatively affected by increasing values of μ_0 than is LR_1^{1d*}. The empirical size of GLS_1^{1d*} approaches zero for increasing values of μ_0. This effect, which starts to become visible for $\mu_0 = 20$, tends to be stronger the larger the dimension of the system when $n \geq 3$. For $n = 2$, the situation is more complex, presumably due to the fact that $n - r_0 = 1$ here. For example, for Cases 1 and 2, the rejection frequencies are decreasing for increasing values of μ_0 if $T = 50$ and $T = 100$, while the opposite is true for $T = 200$. What can clearly be seen again from these results is that the finite sample behaviour of GLS_1^{1d*} is far too unreliable to allow it to be recommended for use in applied work, as has previously also been argued by Trenkler (2009).

Again, using only estimates from an restricted VECM, i.e. applying either Algorithm 2 or 2d, yields bootstrap tests with better size properties than the corresponding tests resulting from Algorithm 1d. As regards, the OLS de-trended tests, LR_1^{2*} appears to be generally preferable to LR_1^{2d*}, the former tending to be less conservative than the latter. Accordingly, it also pays here to exclude the deterministic terms from the bootstrap recursion and to use zero initial values. In contrast, GLS_1^{2d*} tends to be less conservative than GLS_1^{2*}. This relative advantage of GLS_1^{2d*} was also observed for the results from KPSW-DGP. Hence, in this particular case one may keep the estimated deterministic components in the bootstrap recursion. However, this is the only case for which we can make this recommendation.

Once again, the choice between Algorithms 1 and 2 appears not to be crucial. We observe some differences between LR_1^{1*} and LR_1^{2*} and between GLS_1^{1*} and GLS_1^{2*} in the smallest sample size considered ($T = 50$). However, these do not affect the relative performance of the tests. Among the OLS de-trended bootstrap tests, LR_1^{1*} and LR_1^{2*} are preferable in terms of empirical size, while GLS_1^{2d*} is the preferred procedure among the GLS de-trended tests. These tests are seen to best reduce the size distortions from which the corresponding asymptotic tests, LR_1 and GLS_1, suffer. Applying bootstrap methods is seen to be particularly beneficial where the asymptotic tests display either excessive upward size distortions, as occurs, for example, for both the LR_1 and GLS_1 tests in the case of the higher dimensional systems considered, or are under-sized as occurs, for example, in the case of the GLS_1 test in bivariate ($n = 2$) VAR processes (see Table 2).

We now turn to the finite sample power properties of the tests for the null hypothesis that the co-integrating rank is zero. In general, there are no significant differences between the power of the OLS de-trended bootstrap tests. It is only for Case 3 with $T = 50$, that LR_0^{1*} and LR_0^{1d*} have higher power than LR_0^{2*} and LR_0^{2d*}, respectively.

Interestingly, the finite sample power of LR_0^{1d*} tends to only slightly decrease as μ_0

increases, other things equal. This may appear surprising given that the rejection frequencies fall with μ_0 when $r = 1$ is tested. It should be kept in mind, however, that under the null hypothesis of $r = 0$ estimates from different models are not combined in the bootstrap recursion. As discussed at the end of Section 4, here there only occurs a mismatch between the initial values and the remainder of the bootstrap sample. Obviously, for this particular DGP this mismatch does not seem to have a strong impact on the finite sample power of LR_0^{1d*}. In order to clarify this point further, we also simulated power results for tests for the null hypothesis of $r = 1$ in the three-dimensional VAR(2) process with two co-integrating relations ($r_0 = 2$) given by:

$$\begin{aligned} y_t &= \mu_0 + \mu_1 t + x_t \\ \Delta x_t &= \alpha\beta' x_{t-1} + \Gamma_1 \Delta x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, I_3), \quad t = 1, ..., 50, \end{aligned}$$

where $\alpha = \left(\begin{smallmatrix} -0.4 & -0.4 & 0 \\ 0 & -0.6 & 0 \end{smallmatrix}\right)'$, $\beta = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)'$ and $\Gamma_1 = \xi I_3$, with $\xi = 0.5$ as before. Here, for example, the empirical rejection frequencies of the LR_1^{1d*} test (for the null of $r = 1$) are 0.271, 0.252, 0.223, 0.201, 0.184, and 0.178 for $\mu_0 = 0$, 5, 20, 50, 200, and 1000, respectively. Hence, in this set-up the small sample power clearly depends on μ_0. In contrast, LR_1^{1*}, LR_1^{2*} and LR_1^{2d*} have rejection frequencies of 0.300, 0.334 and 0.321, respectively, regardless of the value of μ_0.

Returning to our main results for the tests of $r = 0$ in DGP (5.2), in contrast to what was observed above for LR_0^{1d*}, it is seen that the small sample power of GLS_0^{1d*} is heavily dependent on the value of μ_0; specifically, it approaches zero, other things equal, as μ_0 increases. Moreover, GLS_0^{1d*} has much smaller power than the other GLS de-trended bootstrap tests for the case of $T = 50$. Otherwise, the powers of the GLS de-trended bootstrap tests procedures do not differ as much as their OLS de-trended counterparts. However, notice that GLS_0^{1d*} has higher power in smaller systems ($n = 2$, $n = 3$) for Case 3 when $T = 200$.

Our results also show that there can be significant differences in power between the asymptotic tests and the corresponding bootstrap tests. To assess the importance of these differences one must also compare the empirical size properties of LR_0 and GLS_0 for a non-co-integrated VAR(2) with those of the bootstrap tests. For DGP (5.2) with α and β set to zero, we found that LR_0 and GLS_0 both display large upward size distortions, in particular for $T = 50$ and larger dimensional systems, while the bootstrap tests have smaller sizes often very close to the nominal level. The size distortions of LR_0 are comparable to those of LR_1 in Case 3 of (5.2). Regarding GLS_0, excessive size distortions are even more pronounced than for GLS_1. To illustrate these findings (full details of which are available on request), for $n = 3$, $T = 50$, and $\mu_0 = 0$, LR_0 and GLS_0 have empirical sizes

of 0.236 and 0.129 respectively, while the empirical rejection frequencies for LR_0^{1*}, LR_0^{1d*} LR_0^{2*}, LR_0^{2d*} are 0.062, 0.057, 0.054, and 0.055, respectively, and for GLS_0^{1*}, GLS_0^{1d*}, GLS_0^{2*}, and GLS_0^{2d*} they are 0.057, 0.015, 0.052, and 0.055, respectively. Accordingly, the higher powers of the asymptotic LR_0 and GLS_0 tests would seem to be an artefact of the significant differences in the empirical sizes between the asymptotic and bootstrap tests. Note also that the decrease in power relative to the asymptotic test is generally smaller for the GLS de-trended bootstrap tests than it is for the corresponding OLS de-trended tests (consistent with the smaller size distortions seen in GLS_0 compared to LR_0) with the obvious exception of GLS_0^{1d*} test when the level term, μ_0, is non-zero.

Overall, taking both size and power results into consideration, LR_r^{1*} and GLS_r^{2d*} are respectively the best performing OLS and GLS de-trended bootstrap tests. In a couple of experiments, the OLS de-trended bootstrap procedures have higher finite sample power than the corresponding GLS de-trended tests, while there is no situation in which a GLS de-trended bootstrap test dominates the corresponding OLS de-trended test. Moreover, LR_r^{1*} has, on average, better size properties than GLS_r^{2d*}. Therefore, LR_r^{1*} is the bootstrap test that overall seems to be preferred, given our simulation results.

To summarise, excluding deterministic components from the bootstrap recursion and setting the initial values to zero is nearly always beneficial. It is only for the GLS de-trended tests of Trenkler (2009) (where all of the parameter estimates used in the recursion are obtained from an estimated restricted VECM) that including the estimated deterministic component and using the empirical initial values in the bootstrap recursion scheme appears superior. For both OLS and GLS de-trended tests the bootstrap recursion outlined in Swensen (2006) should be avoided since in both cases it delivers bootstrap tests which are not invariant to the level of the deterministic component. The simulations given here show that the impact on the finite sample behaviour of these tests can indeed be quite large.

6 Conclusions

We have investigated the role of deterministic components and initial values in bootstrap likelihood ratio type tests of co-integration rank, comparing a number of bootstrap procedures that have been proposed in the recent literature some of which include estimated deterministic components and non-zero initial values in the bootstrap recursion while others do the opposite. In the case of OLS de-trended tests, our findings suggest that it is preferable to take the computationally simpler approach of not including estimated deterministic components in the bootstrap recursion and setting the initial values of the

bootstrap recursion to zero. For GLS de-trended tests this approach again works well, although slightly better results can be obtained where all of the parameter estimates used in the recursion are obtained from an estimated restricted VECM, as in Trenkler (2009). Here including the estimated deterministic component and empirical initial values in the bootstrap recursion seems preferable. For both OLS and GLS de-trended tests we recommend against the use of the bootstrap recursion outlined in Swensen (2006) since the resulting bootstrap tests are not invariant to the level of the deterministic component. Taking into account both OLS and GLS de-trended tests, overall we recommend the use of the OLS de-trended bootstrap LR_r^{1*} trace test which sets the deterministic component and initial values to zero in the bootstrap recursion.

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Table 1. Rejection Frequencies of Tests for KPSW-DGP (5.1) with True Rank $r_0 = 2$, VAR Order $p = 2$, Significance Level 0.05

	Panel A: $H_0 : r = 2$			Panel B: $H_0 : r = 1$			Panel C: $H_0 : r = 0$		
	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$
LR_r	0.0040	0.0088	0.0334	0.0374	0.0922	0.3274	0.2792	0.5894	0.9916
LR_r^{1*}	0.0080	0.0156	0.0382	0.0194	0.0742	0.2942	0.1062	0.4485	0.9878
LR_r^{2*}	0.0070	0.0136	0.0390	0.0188	0.0746	0.2933	0.1160	0.4653	0.9882
LR_r^{1d*}	0.0002	0.0026	0.0200	0.0060	0.0412	0.2628	0.1012	0.4409	0.9876
LR_r^{2d*}	0.0020	0.0094	0.0342	0.0154	0.0700	0.2905	0.1164	0.4661	0.9876
GLS_r	0.0072	0.0164	0.0312	0.0352	0.0638	0.1900	0.2098	0.5346	0.9480
GLS_r^{1*}	0.0134	0.0226	0.0328	0.0310	0.0658	0.1892	0.1220	0.4595	0.9390
GLS_r^{2*}	0.0124	0.0206	0.0334	0.0308	0.0662	0.1908	0.1268	0.4641	0.9374
GLS_r^{1d*}	0.0733	0.1449	0.1300	0.0000	0.0002	0.0368	0.0000	0.0116	0.4676
GLS_r^{2d*}	0.0292	0.0380	0.0486	0.0396	0.0858	0.2268	0.1276	0.4647	0.9374

Table 2. Rejection Frequencies of Tests for bivariate VAR(2) (5.2) with $\mu_0 = 0$, True Rank $r_0 = 1$, Significance Level 0.05

	Panel A: Case 1 $a_1 = a_2 = -0.4$			Panel B: Case 2 $a_1 = a_2 = -0.1$			Panel C: Case 3 $a_1 = -0.1, a_2 = -0.8$		
	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$
	Part A: $H_0 : r = 1$ (Empirical Size)								
LR_r	0.0584	0.0548	0.0600	0.0222	0.0494	0.0590	0.1018	0.0722	0.0638
LR_r^{1*}	0.0452	0.0458	0.0502	0.0279	0.0458	0.0534	0.0421	0.0360	0.0407
LR_r^{2*}	0.0454	0.0476	0.0508	0.0240	0.0466	0.0534	0.0566	0.0476	0.0497
LR_r^{1d*}	0.0350	0.0454	0.0502	0.0112	0.0340	0.0518	0.0318	0.0240	0.0389
LR_r^{2d*}	0.0426	0.0478	0.0510	0.0134	0.0398	0.0528	0.0512	0.0467	0.0493
GLS_r	0.0236	0.0264	0.0320	0.0112	0.0112	0.0130	0.0136	0.0128	0.0232
GLS_r^{1*}	0.0402	0.0422	0.0432	0.0239	0.0260	0.0188	0.0233	0.0323	0.0540
GLS_r^{2*}	0.0398	0.0424	0.0434	0.0236	0.0224	0.0194	0.0340	0.0295	0.0436
GLS_r^{1d*}	0.1260	0.0916	0.0586	0.0860	0.0956	0.0818	0.0883	0.1017	0.0877
GLS_r^{2d*}	0.0430	0.0470	0.0474	0.0322	0.0332	0.0252	0.0372	0.0292	0.0268
	Part B: $H_0 : r = 0$ (Empirical Power)								
LR_r	0.8656	1.0000	1.0000	0.2424	0.5782	0.9896	0.9770	1.0000	1.0000
LR_r^{1*}	0.7323	1.0000	1.0000	0.1320	0.4692	0.9822	0.9292	1.0000	1.0000
LR_r^{2*}	0.7444	1.0000	1.0000	0.1254	0.4594	0.9802	0.8808	1.0000	1.0000
LR_r^{1d*}	0.7203	1.0000	1.0000	0.1282	0.4688	0.9820	0.9233	1.0000	1.0000
LR_r^{2d*}	0.7448	1.0000	1.0000	0.1240	0.4584	0.9798	0.8792	1.0000	1.0000
GLS_r	0.6632	0.9366	0.9950	0.1766	0.4982	0.8892	0.5452	0.6052	0.6210
GLS_r^{1*}	0.6944	0.9602	0.9986	0.1426	0.4674	0.8802	0.5077	0.5918	0.6126
GLS_r^{2*}	0.6958	0.9598	0.9986	0.1416	0.4652	0.8778	0.4889	0.5924	0.6148
GLS_r^{1d*}	0.6944	0.9602	0.9986	0.1048	0.4748	0.8816	0.3652	0.5886	0.6206
GLS_r^{2d*}	0.6958	0.9598	0.9986	0.1422	0.4704	0.8830	0.5440	0.6568	0.6934

Table 3. Rejection Frequencies of Tests for bivariate VAR(2) (5.2), True Rank $r_0 = 1$, Significance Level 0.05

	Panel A: Case 1 $a_1 = a_2 = -0.4$			Panel B: Case 2 $a_1 = a_2 = -0.1$			Panel C: Case 3 $a_1 = -0.1, a_2 = -0.8$		
	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$
	Part A: LR_r^{1d*} with $H_0 : r = 1$ (Empirical Size)								
$\mu_0 = 0$	0.0350	0.0454	0.0502	0.0112	0.0340	0.0518	0.0318	0.0240	0.0389
$\mu_0 = 5$	0.0348	0.0464	0.0502	0.0108	0.0346	0.0520	0.0316	0.0240	0.0393
$\mu_0 = 20$	0.0322	0.0452	0.0508	0.0080	0.0338	0.0516	0.0291	0.0231	0.0373
$\mu_0 = 50$	0.0292	0.0442	0.0498	0.0054	0.0294	0.0500	0.0269	0.0203	0.0347
$\mu_0 = 200$	0.0260	0.0392	0.0502	0.0030	0.0230	0.0474	0.0249	0.0192	0.0333
$\mu_0 = 1000$	0.0246	0.0362	0.0466	0.0026	0.0190	0.0416	0.0249	0.0190	0.0311
	Part B: GLS_r^{1d*} with $H_0 : r = 1$ (Empirical Size)								
$\mu_0 = 0$	0.1260	0.0916	0.0586	0.0860	0.0956	0.0818	0.0883	0.1017	0.0877
$\mu_0 = 5$	0.1092	0.1450	0.0988	0.0880	0.0960	0.0848	0.0856	0.1003	0.0923
$\mu_0 = 20$	0.1046	0.1500	0.1006	0.0876	0.0998	0.0814	0.0933	0.1311	0.1066
$\mu_0 = 50$	0.0882	0.1534	0.1060	0.0766	0.1098	0.0842	0.1002	0.1898	0.1442
$\mu_0 = 200$	0.0498	0.1304	0.1682	0.0355	0.1328	0.0848	0.0894	0.1883	0.1737
$\mu_0 = 1000$	0.0216	0.0458	0.2336	0.0132	0.0574	0.1636	0.0861	0.1920	0.1608
	Part C: LR_r^{1d*} with $H_0 : r = 0$ (Empirical Power)								
$\mu_0 = 0$	0.7203	1.0000	1.0000	0.1282	0.4688	0.9820	0.9233	1.0000	1.0000
$\mu_0 = 5$	0.7131	1.0000	1.0000	0.1254	0.4638	0.9814	0.9173	1.0000	1.0000
$\mu_0 = 20$	0.7069	1.0000	1.0000	0.1214	0.4590	0.9810	0.9147	1.0000	1.0000
$\mu_0 = 50$	0.7013	0.9998	1.0000	0.1188	0.4534	0.9810	0.9143	1.0000	1.0000
$\mu_0 = 200$	0.6955	0.9998	1.0000	0.1174	0.4488	0.9800	0.9135	1.0000	1.0000
$\mu_0 = 1000$	0.6961	0.9998	1.0000	0.1156	0.4458	0.9786	0.9124	1.0000	1.0000
	Part D: GLS_r^{1d*} with $H_0 : r = 0$ (Empirical Power)								
$\mu_0 = 0$	0.6944	0.9602	0.9986	0.1048	0.4748	0.8816	0.3652	0.5886	0.6206
$\mu_0 = 5$	0.2165	0.9122	0.9950	0.0904	0.4790	0.8854	0.1625	0.5120	0.6168
$\mu_0 = 20$	0.0002	0.1228	0.9568	0.0124	0.3462	0.8900	0.0014	0.0174	0.2948
$\mu_0 = 50$	0.0000	0.0000	0.1784	0.0000	0.0320	0.7968	0.0002	0.0000	0.0046
$\mu_0 = 200$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000
$\mu_0 = 1000$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4. Rejection Frequencies of Tests for trivariate VAR(2) (5.2) with $\mu_0 = 0$, True Rank $r_0 = 1$, Significance Level 0.05

	Panel A: Case 1 $a_1 = a_2 = -0.4$			Panel B: Case 2 $a_1 = a_2 = -0.1$			Panel C: Case 3 $a_1 = -0.1, a_2 = -0.8$		
	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$
	Part A: $H_0 : r = 1$ (Empirical Size)								
LR_r	0.1106	0.0776	0.0650	0.0534	0.0556	0.0698	0.2356	0.1296	0.0900
LR_r^{1*}	0.0482	0.0488	0.0510	0.0249	0.0382	0.0518	0.0665	0.0434	0.0444
LR_r^{2*}	0.0444	0.0492	0.0496	0.0170	0.0328	0.0504	0.0553	0.0517	0.0537
LR_r^{1d*}	0.0342	0.0466	0.0496	0.0104	0.0234	0.0464	0.0438	0.0398	0.0411
LR_r^{2d*}	0.0398	0.0482	0.0494	0.0110	0.0262	0.0480	0.0519	0.0504	0.0544
GLS_r	0.0482	0.0428	0.0422	0.0198	0.0186	0.0234	0.0530	0.0320	0.0280
GLS_r^{1*}	0.0394	0.0388	0.0432	0.0205	0.0244	0.0298	0.0422	0.0313	0.0341
GLS_r^{2*}	0.0398	0.0392	0.0398	0.0188	0.0242	0.0304	0.0403	0.0336	0.0367
GLS_r^{1d*}	0.0176	0.0430	0.0642	0.0084	0.0240	0.0416	0.0068	0.0199	0.0350
GLS_r^{2d*}	0.0552	0.0502	0.0494	0.0234	0.0360	0.0454	0.0471	0.0439	0.0452
	Part B: $H_0 : r = 0$ (Empirical Power)								
LR_r	0.7906	0.9972	1.0000	0.3482	0.4500	0.9190	0.9524	1.0000	1.0000
LR_r^{1*}	0.4319	0.9878	1.0000	0.1046	0.2758	0.8642	0.7510	0.9996	1.0000
LR_r^{2*}	0.4459	0.9884	1.0000	0.0892	0.2674	0.8624	0.6369	0.9986	1.0000
LR_r^{1d*}	0.4122	0.9876	1.0000	0.0978	0.2728	0.8632	0.7377	0.9994	1.0000
LR_r^{2d*}	0.4449	0.9892	1.0000	0.0864	0.2664	0.8624	0.6337	0.9990	1.0000
GLS_r	0.5910	0.9080	0.9926	0.2098	0.3726	0.8210	0.5622	0.6076	0.6220
GLS_r^{1*}	0.4049	0.8676	0.9908	0.1022	0.2818	0.7818	0.4061	0.5492	0.5934
GLS_r^{2*}	0.4153	0.8706	0.9910	0.0950	0.2776	0.7784	0.3549	0.5276	0.5876
GLS_r^{1d*}	0.2366	0.8646	0.9916	0.0440	0.2738	0.7866	0.1834	0.5300	0.6056
GLS_r^{2d*}	0.4297	0.8754	0.9922	0.0990	0.2826	0.7820	0.4064	0.5876	0.6486

Table 5. Rejection Frequencies of Tests for trivariate VAR(2) (5.2), True Rank $r_0 = 1$, Significance Level 0.05

	Panel A: Case 1 $a_1 = a_2 = -0.4$			Panel B: Case 2 $a_1 = a_2 = -0.1$			Panel C: Case 3 $a_1 = -0.1, a_2 = -0.8$		
	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$
	Part A: LR_r^{1d*} with $H_0 : r = 1$ (Empirical Size)								
$\mu_0 = 0$	0.0342	0.0466	0.0496	0.0104	0.0234	0.0464	0.0438	0.0398	0.0411
$\mu_0 = 5$	0.0328	0.0478	0.0492	0.0102	0.0236	0.0468	0.0446	0.0410	0.0417
$\mu_0 = 20$	0.0306	0.0458	0.0492	0.0086	0.0238	0.0454	0.0403	0.0363	0.0395
$\mu_0 = 50$	0.0270	0.0448	0.0486	0.0062	0.0200	0.0434	0.0365	0.0349	0.0377
$\mu_0 = 200$	0.0230	0.0404	0.0452	0.0044	0.0160	0.0386	0.0324	0.0322	0.0343
$\mu_0 = 1000$	0.0212	0.0380	0.0434	0.0034	0.0120	0.0348	0.0310	0.0316	0.0327
	Part B: GLS_r^{1d*} with $H_0 : r = 1$ (Empirical Size)								
$\mu_0 = 0$	0.0176	0.0430	0.0642	0.0084	0.0240	0.0416	0.0068	0.0199	0.0350
$\mu_0 = 5$	0.0146	0.0444	0.0644	0.0068	0.0262	0.0426	0.0065	0.0181	0.0343
$\mu_0 = 20$	0.0056	0.0374	0.0648	0.0006	0.0156	0.0386	0.0014	0.0128	0.0228
$\mu_0 = 50$	0.0018	0.0190	0.0636	0.0000	0.0042	0.0248	0.0005	0.0020	0.0124
$\mu_0 = 200$	0.0002	0.0014	0.0118	0.0000	0.0006	0.0030	0.0000	0.0000	0.0018
$\mu_0 = 1000$	0.0000	0.0002	0.0002	0.0000	0.0000	0.0002	0.0000	0.0004	0.0000
	Part C: LR_r^{1d*} with $H_0 : r = 0$ (Empirical Power)								
$\mu_0 = 0$	0.4122	0.9876	1.0000	0.0978	0.2728	0.8632	0.7377	0.9994	1.0000
$\mu_0 = 5$	0.4108	0.9878	1.0000	0.0972	0.2732	0.8656	0.7381	0.9996	1.0000
$\mu_0 = 20$	0.3996	0.9872	1.0000	0.0946	0.2694	0.8624	0.7321	0.9994	1.0000
$\mu_0 = 50$	0.3968	0.9858	1.0000	0.0926	0.2694	0.8606	0.7279	0.9996	1.0000
$\mu_0 = 200$	0.3862	0.9856	1.0000	0.0890	0.2630	0.8584	0.7197	0.9994	1.0000
$\mu_0 = 1000$	0.3840	0.9842	1.0000	0.0894	0.2572	0.8554	0.7165	0.9994	1.0000
	Part D: GLS_r^{1d*} with $H_0 : r = 0$ (Empirical Power)								
$\mu_0 = 0$	0.2366	0.8646	0.9916	0.0440	0.2738	0.7866	0.1834	0.5300	0.6056
$\mu_0 = 5$	0.0897	0.8340	0.9916	0.0366	0.2726	0.7902	0.0842	0.4674	0.5980
$\mu_0 = 20$	0.0006	0.0624	0.9478	0.0070	0.1532	0.7958	0.0014	0.0166	0.3164
$\mu_0 = 50$	0.0002	0.0000	0.1286	0.0002	0.0108	0.6308	0.0006	0.0004	0.0048
$\mu_0 = 200$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000
$\mu_0 = 1000$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 6. Rejection Frequencies of Tests for four-dimensional VAR(2) (5.2) with $\mu_0 = 0$, True Rank $r_0 = 1$, Significance Level 0.05, $R = 5000$

	Panel A: Case 1 $a_1 = a_2 = -0.4$			Panel B: Case 2 $a_1 = a_2 = -0.1$			Panel C: Case 3 $a_1 = -0.1, a_2 = -0.8$		
	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$
	Part A: $H_0 : r = 1$ (Empirical Size)								
LR_r	0.2132	0.1262	0.0778	0.0846	0.0652	0.0562	0.4254	0.2056	0.1112
LR_r^{1*}	0.0410	0.0526	0.0502	0.0332	0.0410	0.0418	0.0838	0.0509	0.0422
LR_r^{2*}	0.0322	0.0496	0.0504	0.0294	0.0390	0.0446	0.0524	0.0561	0.0471
LR_r^{1d*}	0.0272	0.0492	0.0500	0.0064	0.0396	0.0652	0.0580	0.0434	0.0410
LR_r^{2d*}	0.0286	0.0486	0.0494	0.0442	0.0532	0.0520	0.0492	0.0545	0.0473
GLS_r	0.0846	0.0652	0.0562	0.0492	0.0272	0.0302	0.1320	0.0580	0.0344
GLS_r^{1*}	0.0332	0.0410	0.0418	0.0150	0.0216	0.0306	0.0472	0.0324	0.0247
GLS_r^{2*}	0.0294	0.0390	0.0446	0.0126	0.0186	0.0320	0.0343	0.0360	0.0274
GLS_r^{1d*}	0.0064	0.0396	0.0652	0.0024	0.0138	0.0384	0.0048	0.0118	0.0190
GLS_r^{2d*}	0.0442	0.0532	0.0520	0.0196	0.0272	0.0454	0.0430	0.0475	0.0378
	Part B: $H_0 : r = 0$ (Empirical Power)								
LR_r	0.8116	0.9882	1.0000	0.5408	0.4630	0.8164	0.9590	0.9994	1.0000
LR_r^{1*}	0.2833	0.9102	1.0000	0.1012	0.1914	0.6786	0.6023	0.9942	1.0000
LR_r^{2*}	0.2745	0.9112	1.0000	0.0798	0.1774	0.6714	0.4310	0.9876	1.0000
LR_r^{1d*}	0.2662	0.9062	1.0000	0.0974	0.1892	0.6766	0.5870	0.9938	1.0000
LR_r^{2d*}	0.2733	0.9100	1.0000	0.0796	0.1792	0.6718	0.4290	0.9874	1.0000
GLS_r	0.6072	0.8804	0.9912	0.3140	0.3388	0.7224	0.6218	0.6196	0.6050
GLS_r^{1*}	0.2677	0.7818	0.9860	0.0830	0.1780	0.6364	0.3095	0.4968	0.5564
GLS_r^{2*}	0.2633	0.7858	0.9866	0.0696	0.1742	0.6304	0.2451	0.4612	0.5408
GLS_r^{1d*}	0.0975	0.7540	0.9880	0.0238	0.1644	0.6464	0.0868	0.4546	0.5658
GLS_r^{2d*}	0.2777	0.7956	0.9872	0.0746	0.1768	0.6322	0.2898	0.5150	0.6044

Table 7. Rejection Frequencies of Tests for four-dimensional VAR(2) (5.2), True Rank $r_0 = 1$, Significance Level 0.05

	Panel A: Case 1 $a_1 = a_2 = -0.4$			Panel B: Case 2 $a_1 = a_2 = -0.1$			Panel C: Case 3 $a_1 = -0.1, a_2 = -0.8$		
	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$
	Part A: LR_r^{1d*} with $H_0 : r = 1$ (Empirical Size)								
$\mu_0 = 0$	0.0272	0.0492	0.0500	0.0092	0.0198	0.0442	0.0580	0.0434	0.0410
$\mu_0 = 5$	0.0262	0.0482	0.0504	0.0100	0.0204	0.0432	0.0588	0.0437	0.0410
$\mu_0 = 20$	0.0226	0.0476	0.0502	0.0078	0.0184	0.0424	0.0551	0.0446	0.0412
$\mu_0 = 50$	0.0190	0.0456	0.0488	0.0058	0.0138	0.0386	0.0503	0.0416	0.0404
$\mu_0 = 200$	0.0166	0.0408	0.0456	0.0050	0.0116	0.0342	0.0452	0.0371	0.0356
$\mu_0 = 1000$	0.0162	0.0372	0.0434	0.0032	0.0108	0.0298	0.0434	0.0337	0.0344
	Part B: GLS_r^{1d*} with $H_0 : r = 1$ (Empirical Size)								
$\mu_0 = 0$	0.0064	0.0396	0.0652	0.0024	0.0138	0.0384	0.0048	0.0118	0.0190
$\mu_0 = 5$	0.0048	0.0370	0.0648	0.0028	0.0146	0.0380	0.0029	0.0100	0.0184
$\mu_0 = 20$	0.0014	0.0262	0.0646	0.0000	0.0054	0.0294	0.0011	0.0070	0.0192
$\mu_0 = 50$	0.0000	0.0072	0.0498	0.0000	0.0010	0.0160	0.0013	0.0016	0.0061
$\mu_0 = 200$	0.0000	0.0004	0.0046	0.0000	0.0002	0.0016	0.0019	0.0007	0.0002
$\mu_0 = 1000$	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0019	0.0009	0.0000
	Part C: LR_r^{1d*} with $H_0 : r = 0$ (Empirical Power)								
$\mu_0 = 0$	0.2662	0.9062	1.0000	0.0974	0.1892	0.6766	0.5870	0.9938	1.0000
$\mu_0 = 5$	0.2656	0.9062	1.0000	0.0960	0.1878	0.6788	0.5798	0.9938	1.0000
$\mu_0 = 20$	0.2604	0.9056	1.0000	0.0954	0.1874	0.6766	0.5780	0.9934	1.0000
$\mu_0 = 50$	0.2520	0.9028	1.0000	0.0922	0.1850	0.6722	0.5730	0.9930	1.0000
$\mu_0 = 200$	0.2493	0.8972	1.0000	0.0900	0.1816	0.6702	0.5634	0.9926	1.0000
$\mu_0 = 1000$	0.2425	0.8940	1.0000	0.0930	0.1802	0.6644	0.5604	0.9924	1.0000
	Part D: GLS_r^{1d*} with $H_0 : r = 0$ (Empirical Power)								
$\mu_0 = 0$	0.0975	0.7540	0.9880	0.0238	0.1644	0.6464	0.0868	0.4546	0.5658
$\mu_0 = 5$	0.0416	0.7012	0.9882	0.0190	0.1618	0.6518	0.0460	0.3926	0.5652
$\mu_0 = 20$	0.0008	0.0416	0.9468	0.0028	0.0788	0.6482	0.0032	0.0188	0.3166
$\mu_0 = 50$	0.0000	0.0000	0.0940	0.0000	0.0044	0.4434	0.0008	0.0004	0.0062
$\mu_0 = 200$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000
$\mu_0 = 1000$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 8. Rejection Frequencies of Tests for five-dimensional VAR(2) (5.2) with $\mu_0 = 0$, True Rank $r_0 = 1$, Significance Level 0.05

	Panel A: Case 1 $a_1 = a_2 = -0.4$			Panel B: Case 2 $a_1 = a_2 = -0.1$			Panel C: Case 3 $a_1 = -0.1, a_2 = -0.8$		
	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$
	Part A: $H_0 : r = 1$ (Empirical Size)								
LR_r	0.4462	0.2018	0.1022	0.3654	0.1330	0.0958	0.7148	0.3438	0.1512
LR_r^{1*}	0.0439	0.0542	0.0468	0.0245	0.0264	0.0382	0.0969	0.0645	0.0466
LR_r^{2*}	0.0276	0.0466	0.0456	0.0116	0.0214	0.0352	0.0444	0.0613	0.0506
LR_r^{1d*}	0.0281	0.0496	0.0448	0.0112	0.0182	0.0334	0.0675	0.0555	0.0434
LR_r^{2d*}	0.0236	0.0454	0.0438	0.0096	0.0180	0.0338	0.0389	0.0594	0.0513
GLS_r	0.1886	0.1062	0.0646	0.1414	0.0508	0.0406	0.3108	0.1084	0.0526
GLS_r^{1*}	0.0347	0.0428	0.0372	0.0203	0.0206	0.0302	0.0533	0.0385	0.0284
GLS_r^{2*}	0.0272	0.0392	0.0356	0.0134	0.0196	0.0312	0.0345	0.0395	0.0306
GLS_r^{1d*}	0.0056	0.0310	0.0574	0.0020	0.0096	0.0334	0.0041	0.0121	0.0195
GLS_r^{2d*}	0.0414	0.0550	0.0436	0.0208	0.0268	0.0424	0.0484	0.0513	0.0464
	Part B: $H_0 : r = 1$ (Empirical Power)								
LR_r	0.9072	0.9772	1.0000	0.7870	0.5408	0.7408	0.9826	0.9994	1.0000
LR_r^{1*}	0.2249	0.7824	1.0000	0.1249	0.1620	0.4976	0.5095	0.9680	1.0000
LR_r^{2*}	0.1918	0.7822	1.0000	0.0820	0.1388	0.4836	0.3234	0.9234	1.0000
LR_r^{1d*}	0.2093	0.7764	1.0000	0.1134	0.1570	0.4952	0.4926	0.9662	1.0000
LR_r^{2d*}	0.1922	0.7798	1.0000	0.0820	0.1354	0.4836	0.3198	0.9232	1.0000
GLS_r	0.7336	0.8816	0.9910	0.5466	0.3986	0.6592	0.7890	0.6756	0.6292
GLS_r^{1*}	0.1933	0.6762	0.9782	0.0993	0.1410	0.4716	0.2866	0.4681	0.5470
GLS_r^{2*}	0.1696	0.6746	0.9784	0.0696	0.1236	0.4692	0.1889	0.4163	0.5187
GLS_r^{1d*}	0.0485	0.6124	0.9814	0.0172	0.1148	0.4978	0.0520	0.3989	0.5628
GLS_r^{2d*}	0.1878	0.6894	0.9794	0.0762	0.1278	0.4766	0.2384	0.4723	0.5843

Table 9. Rejection Frequencies of Tests for five-dimensional VAR(2) (5.2), True Rank $r_0 = 1$, Significance Level 0.05

	Panel A: Case 1 $a_1 = a_2 = -0.4$			Panel B: Case 2 $a_1 = a_2 = -0.1$			Panel C: Case 3 $a_1 = -0.1, a_2 = -0.8$		
	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$	$T = 50$	$T = 100$	$T = 200$
	Part A: LR_r^{1d*} with $H_0 : r = 1$ (Empirical Size)								
$\mu_0 = 0$	0.0281	0.0496	0.0448	0.0112	0.0182	0.0334	0.0675	0.0555	0.0434
$\mu_0 = 5$	0.0267	0.0498	0.0450	0.0110	0.0182	0.0334	0.0650	0.0541	0.0440
$\mu_0 = 20$	0.0224	0.0472	0.0454	0.0086	0.0170	0.0332	0.0574	0.0508	0.0438
$\mu_0 = 50$	0.0212	0.0446	0.0446	0.0072	0.0156	0.0330	0.0495	0.0476	0.0418
$\mu_0 = 200$	0.0168	0.0372	0.0424	0.0062	0.0114	0.0282	0.0427	0.0443	0.0398
$\mu_0 = 1000$	0.0162	0.0356	0.0406	0.0062	0.0092	0.0242	0.0411	0.0418	0.0390
	Part B: GLS_r^{1d*} with $H_0 : r = 1$ (Empirical Size)								
$\mu_0 = 0$	0.0056	0.0310	0.0574	0.0020	0.0096	0.0334	0.0041	0.0121	0.0195
$\mu_0 = 5$	0.0030	0.0288	0.0588	0.0012	0.0090	0.0318	0.0041	0.0109	0.0201
$\mu_0 = 20$	0.0000	0.0178	0.0574	0.0002	0.0048	0.0254	0.0028	0.0058	0.0162
$\mu_0 = 50$	0.0000	0.0038	0.0334	0.0000	0.0006	0.0102	0.0025	0.0028	0.0059
$\mu_0 = 200$	0.0000	0.0000	0.0012	0.0000	0.0000	0.0004	0.0038	0.0019	0.0004
$\mu_0 = 1000$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0063	0.0039	0.0004
	Part C: LR_r^{1d*} with $H_0 : r = 0$ (Empirical Power)								
$\mu_0 = 0$	0.2093	0.7764	1.0000	0.1134	0.1570	0.4952	0.4926	0.9662	1.0000
$\mu_0 = 5$	0.2097	0.7760	1.0000	0.1144	0.1562	0.4944	0.4916	0.9648	1.0000
$\mu_0 = 20$	0.2051	0.7728	1.0000	0.1116	0.1556	0.4938	0.4856	0.9658	1.0000
$\mu_0 = 50$	0.2057	0.7684	1.0000	0.1080	0.1538	0.4922	0.4831	0.9646	1.0000
$\mu_0 = 200$	0.1979	0.7628	1.0000	0.1076	0.1494	0.4886	0.4747	0.9636	1.0000
$\mu_0 = 1000$	0.1949	0.7580	1.0000	0.1058	0.1462	0.4874	0.4689	0.9624	1.0000
	Part C: GLS_r^{1d*} with $H_0 : r = 0$ (Empirical Power)								
$\mu_0 = 0$	0.0485	0.6124	0.9814	0.0172	0.1148	0.4978	0.0520	0.3989	0.5628
$\mu_0 = 5$	0.0262	0.5460	0.9818	0.0144	0.1088	0.4992	0.0293	0.3443	0.5562
$\mu_0 = 20$	0.0014	0.0296	0.9248	0.0012	0.0432	0.4976	0.0034	0.0206	0.3400
$\mu_0 = 50$	0.0000	0.0000	0.0642	0.0002	0.0024	0.2718	0.0008	0.0010	0.0070
$\mu_0 = 200$	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000	0.0004	0.0002	0.0000
$\mu_0 = 1000$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000